ALGEBRAIC GEOMETRY - FINAL EXAMINATION

Attempt all questions - 10 a.m. to 1 p.m., 9th May 2012 - Total Marks:50

In all questions below, the field k is algebraically closed, unless otherwise mentioned.

(1) Let k^2 be the affine plane and let $\mathcal{O}$ be the sheaf of regular functions on k^2. Let $U = k^2 - \{(0,0)\}$ be the complement of the origin, considered as open subset of k^2. Prove that $\mathcal{O}(U) \cong k[X,Y]$, and the restriction map $\mathcal{O}(k^2) \to \mathcal{O}(U)$ is the identity map on $k[X,Y]$. Using this, prove that $(U, \mathcal{O}_U)$ cannot be isomorphic to an affine variety (here $\mathcal{O}_U$ denotes the restriction of $\mathcal{O}$ to U). (6+6 marks)

(2) Let $\mathbb{P}^n(k)$ be the n dimensional projective space over k, and let $\mathcal{O}$ denote the sheaf of regular functions on $\mathbb{P}^n(k)$. Let f be any homogeneous polynomial of positive degree. Describe $\mathcal{O}(D_+(f))$ and verify that $\mathcal{O}$ satisfies the sheaf conditions on such basic open sets. Prove that $(\mathbb{P}^n(k), \mathcal{O})$ is an algebraic variety. (8+8 marks)

(3) Prove that a morphism $\phi : X \to Y$ between two algebraic varieties is an isomorphism, if and only if, the map ϕ is a homeomorphism and the induced maps of stalks $\phi^* : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ is an isomorphism of local rings for all $x \in X$. (3+5 marks)

(4) Let $\phi : \mathbb{P}^1 \to \mathbb{P}^n$ be the map defined by $\phi(x : y) = (x^n : x^{n-1}y : x^{n-2}y^2 : \ldots : xy^{n-1} : y^n)$. Let C be the image of ϕ. Prove that C is a closed subset of $\mathbb{P}^n$ and $\phi : \mathbb{P}^1 \to C$ is an isomorphism of algebraic varieties. (7+7 marks)